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MOTION OF AN INITIALLY POINT VORTEX IN A FLOW OF VISCOUS FLUID*

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Asymptotic equations and their solutions, describing the motion and diffusion of an initially point vortex in an arbitrary flow of a low viscosity fluid, are obtained. A model example is used to demonstrate the possibility of the occurrence of stochastic modes of motion of the vortex.

A scheme due to Chorin has recently been finding favour when dealing numerically with the systems of point vortices in an ideal fluid. In the scheme the vortices move as if they were in an ideal fluid, and diffuse as if they were in a stationary viscous fluid (without taking into account the fact that in the latter case the flow would become vortical everywhere) /1/. Attempts were made to justify such a scheme, even though they referred largely to the problem of choosing the best forms, in a certain sense, of approximating the vorticity /2/ and not to establishing the degree of approximation to the solutions in the exact formulation. An example is known /3/ of a scheme for computing the dynamics of a system of vortices in a low-viscosity fluid involving a description in terms of the distribution functions, realized numerically with help of a "viscous" random walk in the spirit of /1/.

In the case when similar numerical experiments were aimed at, say, constructing a statistical flow chart /4-7/ where the question arises of the validity of using similar schemes to describe small-scale motions responsible for viscous dissipation, the problem of deriving a solution for the motion of the vortices based on an exact formulation within the framework of the Navier-Stokes equations, is quite important.

Only two classical exact solutions of the Navier-Stokes equations of this type are known, which describe the diffusion of an initial point vortex and of a circular domain of constant vorticity. Attempts to construct approximate solutions describing the motion and diffusion of a vortex of small cross-section in a flow of viscous fluid have been few, and mostly of a qualitative character (e.g. the motion of a vortex pair /8/). Recently, examples have appeared of the determination of the evolution of finite vortex domains by integrating the Navier-Stokes equations numerically**.(**Getling A.V. On the interaction of vortices of finite dimension in a viscous fluid. Preprint No.87-016. Moscow, Scientific Research Institute, of Nuclear Physics, Moscow State Univ., 1987.) Such approaches are however procedurally difficult to apply to the problem formulated above.

1. Let a point vortex of intensity Γ appear at the instant $t=0$ in an arbitrary plane unbounded flow of a viscous incompressible fluid with vorticity field $\Omega^0(x, y, t)$, $t<0$. The dynamics of the vorticity are described by the Helmholtz equation

$$D(\Omega)+u\Omega_x+v\Omega_y=0,\quad D\equiv\partial/\partial t-\nu\Delta \tag{1.1}$$

Let us represent, for $t>0$, the vorticity and velocity fields in the form $u=u^0+u^1$, $v=v^0+v^1$, $\Omega=v_x-u_y=\Omega^0+\Omega^1(x', y', t)$, $x'=x-X$, $y'=y-Y$ where $X=X(t)$, $Y=Y(t)$ is the trajectory of the centre of the vortex, determined by the condition for an extremum of the vorticity

$$\Omega_x(X, Y, t)\equiv\bar{\Omega}_x=0,\quad \Omega_y(X, Y, t)\equiv\bar{\Omega}_y=0 \tag{1.2}$$

The perturbation in the main field connected with the presence of a vortex within the flow is described, as follows from (1.1), by the equation (here we have taken into account the fact that the main field satisfies the Helmholtz equation, and a dot denotes differentiation with respect to t)

$$D'(\Omega^1)+(u^0+u^1-X^{\cdot})\,\Omega_{x'}{}^1+(v^0+v^1-Y^{\cdot})\,\Omega_{y'}{}^1+u^1\Omega_x{}^0+ v^1\Omega_y{}^0=0. \tag{1.3}$$

with the initial condition

$$\Omega^1(t=0)=\Gamma\delta(x')\,\delta(y') \tag{1.4}$$

We shall assume that the viscosity ν is low enough for the vortex to retain, within the characteristic diffusion time θ, the fundamental part of the vorticity within the neighbourhood

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of the point $x' = y' = 0$ of size $\varepsilon = \sqrt{\nu\theta}$, and we shall introduce the inner variables $\xi = \varepsilon^{-1}x'$, $\eta = \varepsilon^{-1}y'$, $\tau = \theta^{-1}t$. In order to avoid bulky expressions, we shall also assume that the characteristic magnitudes of the basic flow are equal to unity.

By virtue of condition (1.4), which, in the new variables, takes the form

$$\Omega^1(\xi, \eta, \tau = 0) = \Gamma\varepsilon^{-2}\delta(\xi)\,\delta(\eta)$$

we shall seek the inner solution of Eq.(1.3), with conditions (1.2), in the form of asymptotic expansions

$$\Omega^1 = \varepsilon^{-2}(\Omega_0{}^1 + \varepsilon\Omega_1{}^1 + \ldots), \quad u^1 = \varepsilon^{-1}(u_0{}^1 + \varepsilon u_1{}^1 + \ldots), \quad X = X_0 + \varepsilon X_1 + \ldots$$

Substitution of these expansions leads to the following equations and conditions in the first two approximations (the operator D is written in the variables ξ, η):

$$(\bar{u}^0 - X_0{}^\cdot)\,\Omega_{0\xi}{}^1 + (\bar{v}^0 - Y_0{}^\cdot)\,\Omega_{0\eta}{}^1 = 0 \tag{1.5}$$

$$\theta^{-1}D\Omega_0{}^1 + U\Omega_{0\xi}{}^1 + V\Omega_{0\eta}{}^1 + (\bar{u}^0 - X_0{}^\cdot)\,\Omega_{1\xi}{}^1 + (\bar{v}^0 - Y_0{}^\cdot)\,\Omega_{1\eta}{}^1 = 0 \tag{1.6}$$

$$U = a_{11}(\xi + X_1) + a_{12}(\eta + Y_1) - X_1{}^\cdot$$

$$V = a_{21}(\xi + X_1) + a_{22}(\eta + Y_1) - Y_1{}^\cdot$$

$$\Omega_{0\xi}{}^1(0, 0, \tau) \equiv \bar{\Omega}_{0\xi}{}^1 = 0, \quad \bar{\Omega}_{0\eta}{}^1 = 0 \tag{1.7}$$

$$\Omega_0{}^1(\xi, \eta, 0) = \Gamma\delta(\xi)\,\delta(\eta) \tag{1.8}$$

where $\bar{w} \equiv w(X_0, Y_0)$, $a_{11} = \bar{u}_x{}^\circ$, $a_{12} = \bar{u}_y{}^\circ$, $a_{21} = \bar{v}_x{}^\circ$, $a_{22} = \bar{v}_y{}^\circ$ and $a_{22} = -a_{11}$ by virtue of the equations of continuity.

The solution of Eq.(1.5) is an arbitrary smooth function of the variable $\zeta = (\bar{v}^0 - Y_0{}^\cdot)\,\xi + (\bar{u}^0 - X_0{}^\cdot)\,\eta$, and in accordance with the boundary conditions (1.7) and (1.8) this is possible only when

$$X_0{}^\cdot = \bar{u}^0, \quad Y_0{}^\cdot = \bar{v}^0 \tag{1.9}$$

The relations obtained represent the equations of the trajectory of the vortex in the zeroth approximation.

Let us change to the new variables

$$\tau' = \tau, \quad \xi' = (1 - A_{11})\,\xi - A_{12}\eta - \int_0^t U(0, 0, t)\,dt$$

$$\eta' = -A_{21}\xi + (1 + A_{11})\,\eta - \int_0^t V(0, 0, t)\,dt; \quad A_{ij} = \int_0^t a_{ij}\,dt$$

in which Eq.(1.6) will become

$$\Omega^1_{0\tau'} = \Omega^1_{0\xi'\xi'} + \Omega^1_{0\eta'\eta'}$$

The above equation has a well-known solution satisfying condition (1.8)

$$\Omega_0{}^1 = \frac{\Gamma}{4\pi t}\exp\left(-\frac{\xi'^2 + \eta'^2}{4t}\right) \tag{1.10}$$

Relations (1.7) mean that $\Omega_0{}^1$ has an extremum at the point $\xi' = \eta' = 0$, and this implies that $U(0, 0, t) = 0$, $V(0, 0, t) = 0$. The inner solution obtained using this approach must be matched with the corresponding outer solution /9/. Leaving aside the investigation of specific types of external field $\Omega^0(x, y, t, \varepsilon)$ and their properties as $\varepsilon \to 0$, it is natural to make a fairly general assumption that for the asymptotic form $\varepsilon \to 0$ in the outer representation we have a corresponding inner limit $\lim \Omega^1(\varepsilon\xi, \varepsilon\eta, t, \varepsilon) = 0$ as $\varepsilon \to 0$. This satifies the requirement for matching the outer and inner expansions /9/, apart from exponentially small terms, since by virtue of (1.10) the outer limit $\lim \Omega^1(\varepsilon^{-1}x', \varepsilon^{-1}y', t, \varepsilon) = 0$ as $\varepsilon \to 0$.

A similar class Ω^0 embraces the solutions which vary, as $\varepsilon \to 0$, at least not more slowly than $O(\varepsilon)$, and this is, generally speaking, a fairly weak requirement since the order of viscous terms in the equation for Ω^0 is $O(\varepsilon^2)$. Thus the assumption discussed here makes it unnecessary, for a wide class of the outer flows, to have to construct a detailed asymptotic outer expansion to a first approximation.

Thus the first approximation for the perturbation of the vortex trajectory is described by the system

$$X_1^{\cdot\cdot} - \sigma(t) X_1 = 0, \quad Y_1 = a_{12}^{-1}(X_1^{\cdot} - a_{11}X_1); \qquad (1.11)$$
$$\sigma(t) = a_{11}^2 + a_{12}a_{21}$$

At first sight, the "splitting" of the zeroth approximation (1.9), (1.10) obtained is in fact different from that of Chorin, irrespective of the outer matching. The trajectories are obtained by integrating the velocity fields of the main viscous flow, and the characteristics of the diffusion of vorticity are found to be connected with the characteristics of the main flow. However, in fact, when solutions for perturbed motion are obtained, the viscous properties of the basic flow field do not appear at all in the approximation discussed here and the situation does not change if we regard the main flow as inviscid and the viscosity as "appearing" in small scale phenomena of the order of ε, since the viscous term in the Helmholtz equation is of the order of $O(\varepsilon^2)$. Thus, if we regard the main flow in this sense as inviscid "almost everywhere", then the use of inviscid velocities in (1.9) will lead to an error not greater than $O(\varepsilon^2)$. If at the same time u^0, v^0 are such that $A_{ij}(t)$ are found to be sufficiently small, then the heuristic scheme of Chorin and the scheme (1.9), (1.10) following from the equations, will be similar.

However, the fact that the perturbations of the trajectories of the order of ε (1.11) exist within this approximation, makes all these "splittings" basically different. Taking into account such small scale dynamics may result in the discovery of new qualitative properties of the evolution of vortex systems with viscosity, such as, for example, the possible randomization of the motion of an isolated vortex within the flow when it has well-defined characteristics. Below we shall discuss a model example as an illustration of this position.

2. Let two point vortices of equal strength Γ appear at a distance l from each other at the instant $t < 0$, in an unbounded viscous fluid which is at rest at $t = 0$.

We shall regard the velocity field due to one vortex, described by the classical solution for a fixed vortex in an unbounded viscous fluid as the external u^0, v^0 field for the other vortex, and vice versa. Such an approximation is found within the framework of the assumption used here of matching the inner and outer solutions, and makes it possible for the class Ω^0 containing terms exponential in ε, to retain these terms in the initial approximation without altering the structure of the asymptotic expansions. Then the corresponding dynamic problem in the zeroth approximation (1.9) will be described by the system

$$X_{01}^{\cdot} = (Y_{01} - Y_{02})\, G, \quad Y_{01}^{\cdot} = -(X_{01} - X_{02})\, G$$
$$G = \frac{\Gamma}{2\pi R_{12}^2}\left[1 - \exp\left(-\frac{R_{12}^2}{4\nu t}\right)\right], \quad R_{12}^2 = (X_{01} - X_{02})^2 + (Y_{01} - Y_{02})^2$$
$$X_{02}^{\cdot} = -X_{01}^{\cdot}, \quad Y_{02}^{\cdot} = -Y_{01}^{\cdot}, \quad R_{12}(t = 0) = l$$

whose solution will be the trajectory

$$X_{01} = {}^1/_2 l \cos \omega t = -X_{02}, \quad Y_{01} = {}^1/_2 l \sin \omega t = -Y_{02}$$
$$\omega = 4\Gamma/\pi l^2$$

Without pausing to analyse the law of diffusion in this approximation (1.10), we shall consider the equation describing the perturbation of the trajectory X_1 from (1.11), which can be transformed to the form (a prime denotes differentiation with respect to z)

$$X_1'' + [a(z) + 2q(z)\cos 2z]\, X_1(z) = 0 \qquad (2.1)$$
$$a(\zeta) = -3\cdot 2^{-8}\zeta\,[1 + ({}^5/_{12}\zeta - 1)\, e^{-\zeta}]\, e^{-\zeta}$$
$$q(\zeta) = 2^{-8}\,[2 + (\zeta - 2)\, e^{-\zeta} - (\zeta - {}^5/_4)\,\zeta e^{-2\zeta}]$$
$$\zeta = \alpha/z, \quad \alpha = 2\Gamma/\pi\nu, \quad z = 2\omega t$$

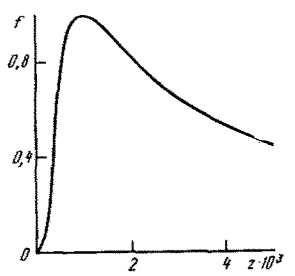
When the viscosity ν is sufficiently small, the characteristic scale of variation of z in the exponential term α, is much larger than the period $\cos 2z$; therefore, the functions $a(z)$ and $q(z)$ in the resulting equation vary slowly and can be regarded, within the framework of the qualitative analysis, as constant. In this case (2.1) will become the well-known Mathieu equation (see e.g. /10/), about which we know, in particular, that its solutions describe, within the specified domains of variation of a and q, the phenomena of parametric excitation and resonance /11/.

We note that for the formal scheme of inviscid basic flow we have the corresponding case of $a = 0,\ q = 2^{-7}$ (by making the change of variable $z = z' + \pi/2$ we reduce Eq.(2.1) to its canonical form with a minus sign in front of q), and in the (a, q) plane the corresponding point P which lies in the region of stable solutions between the neighbouring separatrices $a_0(q)$ and $b_1(q)$ (in the notation of /12/), separating the region of stability from the regions of exponentially increasing oscillations. Thus in the scheme without viscosity the perturbed motion is stable and is described by solutions of the form $X_1(z, a, q) = Ce^{i\beta z}\Phi(z)$ where $\Phi(z)$ is a periodic Mathieu function $\mathrm{se}_\lambda(z), \mathrm{ce}_\lambda(z), \lambda = \sqrt{a}$ if β is a rational quotient, or a quasiperiodic Mathieu function $\mathrm{fe}_\lambda(z), \mathrm{ge}_\lambda(z)$, if β is an irrational number. However, since the point P lies in the immediate vicinity of the separatrix, the situation can change fundamentally for arbitrarily small q, e.g. in the course of numerical computation. Indeed, the distance from the point P to a point lying on the separatrix $a_0(q)$ or $b_1(q)$ can be easily estimated for small q from the equation of the separatrix /10, 12/: $a_0(q) \sim -{}^1/_2 q^2 + O(q^4)$ (and for b_1 in the same manner). It is now sufficient to introduce into the scheme errors in the value of a of the order of 2^{-15}, to arrive at the region of instability either below $a_0(q)$ or above $b_1(q)$, depending on the sign of the error. Besides, the situation may require a more accurate analysis, which is outside the scope of this paper.

When the viscosity is not zero, then, irrespective of the fact that the quantity a is different from zero (even if it is small for sufficiently large α), the position for which $|a_0(q)| > {}^1/_2 q^2$ is attained at finite, though fairly large values of z (the graph of $f(z) = a(z)/a_m$, $a_m = -3\cdot 2^{-8}e^{-1}$ is shown in the figure).

In this case the solution of Eq.(2.1) will have the form $X_1 = Ce^{\mu z}\Phi(z)$, $\mu > 0$ where μ is the characteristic index the method of determining which is well-known /10/, and Φ is a, generally speaking, quasiperiodic function of the type described above.

If we choose, say, $\Gamma = 1$, $\nu = 10^{-4}$, then the value of z at which the mode of unstable oscillations with exponentially increasing amplitude is reached is of the order of 10^3, and this corresponds in real time to about ten rotations of the system of vortices. This time will increase as ν decreases.



The presence of a local exponential instability in Hamiltonian systems with parametric excitation of the type (2.1), ensures the stochastic character of the behaviour of the solution /11/. Systems of the more general type (1.11) will, in all likelihood, also contain a mechanism for the randomization of the solutions, at least under the condition that the function $\sigma(t)$ is periodic or quasiperiodic. This can occur in problems of the dynamics of systems of initially point vortices in a viscous fluid, in flows with rotations or oscillations of the stream, and in other similar flows.

We suggest that the discovery of such a mechanism offers the chance of a fresh look at the heuristic introduction of artificial randomization of trajectories in the course of carrying out numerical experiments with systems of point vortices in an ideal fluid (see e.g. /13/ and the references given there) which obtains a specific argumentation within the framework of a deterministic description.

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COMPUTATION OF ATTACHED FLOW PAST AN AIRFOIL PROFILE AT HIGH REYNOLDS NUMBERS*

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A mathematical model of an attached flow of an incompressible fluid past an airfoil profile at high Reynolds numbers is proposed. The model enables one to determine the effect of viscosity on the magnitude of aerodynamic characteristics. Not only is the displacing action of the turbulent boundary layer and wake on the external flow taken into account, but the solution in the neighbourhood of the trailing edge is also studied, and this makes it possible to formulate a more accurate analogue of the Chaplygin-Zhukovskii condition. Comparison of numerical results with experimental data shows that the accuracy of the results is comparable with that of experiment. The flow past a profile is usually computed by solving a sequence of problems arising when the concept of the Prandtl boundary layer is applied regularly. In this approach the external problems describe the flow of an inviscid fluid past modifications of the profile, which take into account the displacement of the boundary layer and distortion of the wake. Their unique solution satisfies the additional demand of regularity. To a first approximation such a demand is represented by the Chaplygin-Zhukovskii condition. To a second approximation the condition is obtained by analysing the solution of the Navier-Stokes equations near the trailing edge of the profile. In the present paper the analysis is carried out for a profile with a sharp trailing edge, in which the angle between the tangents is not zero.

1. Let us consider the flow of an incompressible fluid past an airfoil profile. We shall regard the segment of the straight line between the leading and trailing edge of length L as the chord of the profile, and the angle α between the direction of the velocity at infinity U_∞ and the chord, as the angle of attack. We shall refer all linear parameters to L, the velocity to U_∞, and the pressure p to the square of the pressure head ρU_∞^2 where ρ is the density of the fluid. We shall place the origin of a Cartesian coordinate system xy at the trailing edge of the profile, and direct the x axis along the bisector of the angle β ($\beta \ll 1$) within it.

We shall consider the solution of the problem of the flow of an ideal incompressible fluid past a profile in the plane $\zeta = r\exp(i\omega)$. The outside of the profile will map onto the outside of the unit circle $|\zeta| = 1$ in this plane, and the trailing edge of the profile will correspond to the point $\zeta = 1$.

We can assume, without loss of generality, that such a mapping can be carried out using the method described in /1/. We shall use the Karman-Trefftz transformation

$$(\xi - \xi_0)/(\xi + \xi_0) = [(z - z_0 - k_1\xi_0)/(z - z_0 + k_1\xi_0)]^{1/k_1}, \quad k_1 = 2 - \beta/\pi$$

to map the profile onto the part of the plane bounded by an almost spherical contour. After this we will seek the coefficients A_j and B_j of the Theodorsen-Garrick transformation

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